GENERALIZATION OF GALIN'S PROBLEM TO FRICTIONAL MATERIALS AND DISCONTINUOUS STRESS FIELDS

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Abstract—An analytical solution for the determination of the form of the plastic zone in the vicinity of a circular hole in inhomogeneous stress fields is discussed. Already in 1946 Galin arrived at a solution to a related problem of an infinite plate with a circular aperture in stretching and bending. Around the aperture, the material behaves perfectly plastically, obeying Tresca's yield criterion. However, in Galin's solution the explicit condition for the general balance of forces in bending has been omitted. This paper presents a correction to Galin's error by means of a modified analytical approach. An extension of the problem, by applying Coulomb's yield criterion in the plastic zone and assuming continuous or discontinuous stress distributions at the elastoplastic interface is also considered. The results are illustrated by some numerical examples and supplemented by an experiment.

1. INTRODUCTION

Cylindrical cavities under hydrostatic loading have been the subject of many studies. In the case of more complex loading, the Finite Element Method for examinations was usually used. Only a few studies were concerned with the problem of non-hydrostatic loading of circular cavities with analytic or semi-analytic methods. Non-hydrostatic homogeneous loading of long cylindrical cavities was considered thoroughly by Detournay and Fairhurst (1987). The object of this paper is to investigate the non-hydrostatic and additionally inhomogeneous loading of long cylindrical cavities in an analytical way for Tresca's material, and in a semi-analytical way for Coulomb's material. Inhomogeneous loading means that the initial stress (or the far-field stress) in the plate under consideration is inhomogeneous.

Examinations of the kinematics of the problem were not performed. Since the stress field and the displacement field in the plastic zone are independent of one another for the cases analysed, a separate examination of the kinematics would be possible.

The presented statical solution is valid only up to a certain amount of deviation of the plastic region from a cylindrical shape. The limits are discussed by Detournay and Fairhurst (1987).

Consider an infinite plate containing a circular hole under stretching and bending. Assume that a plastic zone completely encloses the hole, obeying Coulomb's yield condition, which is an extension of Tresca's condition in the plane strain problem examined.

A solution to the described problem for a biaxially stretched plate under Tresca's yield condition was given by Galin (1946), who found that the elastoplastic interface has an elliptic shape. A solution to the same problem under Coulomb's yield condition was given by Detournay (1986). He found that the shape of the interface in this case reminds one of a compressed ellipse.

Galin (1946) also provided a solution to the problem in the case of inhomogeneous loading by bending or superimposed bending and biaxial stretching. His solution can be shown to be incorrect, because the condition for the general balance of forces has been omitted. It is the object of this paper to correct Galin's error and to extend his solution to Coulomb's yield condition in the plastic zone and to continuous or discontinuous circumferential stresses at the elastoplastic interface.

Boundary conditions at the elastoplastic interface consist of equilibrium conditions and geometrical compatibility conditions between the elastic and the plastic fields. In Galin's
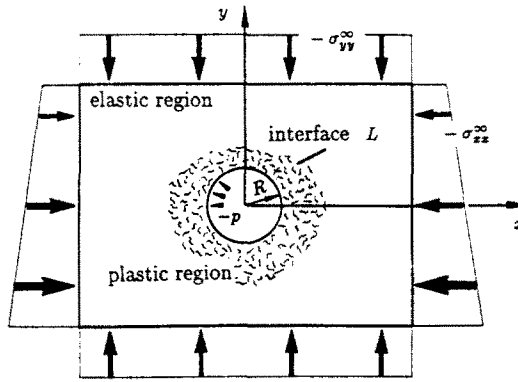
Fig. 1. Problem definition, inhomogeneous loading.

and Detournay's solutions to the problem, continuity of all stresses was required. This condition is in general not necessary and will be replaced by other conditions in this paper. Therefore, equilibrium and compatibility conditions will be considered separately.

2. PROBLEM STATEMENT AND SOLUTION FOR TRESCA'S YIELD CONDITION AND CONTINUOUS STRESS FIELDS

In Section 2 the fundamental equations will be presented and a correction to Galin's solution with respect to the form of the plastic zone around a circular hole in a bent plate will be derived.

2.1. Plastic region

Consider a normal stress σ_{rr} and shear stress $\tau_{r\varphi}$ acting upon the border of a circular hole with radius R:

$$\sigma_{rr} = p; \quad \tau_{r\varphi} = 0. \tag{1}$$

r and φ are polar co-ordinates, and $r = R$ is the surface of the hole. In the case of Tresca's yield criterion, the maximum shear stress is $\tau_{max} = k$. The stress in the plastic zone is axial-symmetric and has to fulfill the yield condition:

$$|\sigma_{rr} - \sigma_{\varphi\varphi}| = \sqrt{(\sigma_{xx} - \sigma_{yy})^2 + 4\tau_{xy}^2} = 2k. \tag{2}$$

x, y is a Cartesian co-ordinate system. The equilibrium conditions are valid both in the elastic zone and in the plastic zone:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0; \quad \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0. \tag{3}$$

In the problem described above, vanishing body forces are assumed and equilibrium in the third direction is identically accomplished.

Assuming that the plastic zone completely encloses the circular hole, Fig. 1, Galin (1946) calculates the stress distribution in the plastic region in terms of a plastic potential U_1:

$$\sigma_{xx} = \frac{\partial^2 U_1}{\partial y^2}; \quad \sigma_{yy} = \frac{\partial^2 U_1}{\partial x^2}; \quad \tau_{xy} = -\frac{\partial^2 U_1}{\partial x \partial y}. \tag{4}$$

Galin has used the solution for the plastic potential U_1:

$$U_1 = kr^2 \ln \frac{r}{R} - \frac{k-p}{2} r^2 \quad \text{with} \quad \sigma_{rr} \leqslant \sigma_{\varphi\varphi} \tag{5.1}$$

$$\tfrac{1}{2}(\sigma_{xx}+\sigma_{yy}) = \frac{k+p}{2} + k \ln \left|\frac{z}{R}\right|; \quad \tfrac{1}{2}(\sigma_{yy}-\sigma_{xx}) + \mathrm{i}\tau_{xy} = k\frac{\bar{z}}{z}. \tag{5.2}$$

The complex representation of stresses used (5.2) is necessary for the next section and will be explained there. The potential function U_1 is a biharmonic function. According to the Laplace operator $\Delta = (\partial^2/\partial x^2 + \partial^2/\partial y^2)$, U_1 satisfies eqn (6):

$$\Delta\Delta U_1 = 0. \tag{6}$$

2.2. Elastic region

The solution for the plane problem in the theory of elasticity can be expressed by two complex functions $\varphi_2(z)$ and $\psi_2(z)$, describing an elastic biharmonic potential U_2 by means of Kolosov's (1909) and Muskhelishvili's (1953) method. In the problem in question an infinite elastic region surrounds the plastic regime. At infinity stresses σ_{xx}^∞ and σ_{yy}^∞ are acting. Beside the equilibrium eqns (3), Hooke's law concerning the linear coupling of stresses and strains and conditions of compatibility has to be applied to the elastic region. Introducing the complex variable $z = x + \mathrm{i}y$, with $\mathrm{i} = \sqrt{-1}$, the stress distribution with these conditions is known to be:

$$(\sigma_{xx}+\sigma_{yy}) = 4\mathscr{R}[\varphi_2'(z)] \tag{7}$$

$$(\sigma_{yy}-\sigma_{xx})+2\mathrm{i}\tau_{xy} = 2[\bar{z}\varphi_2''(z)+\psi_2'(z)] \tag{8}$$

where $\mathscr{R}[\ ]$ means the real part of an expression, $\bar{z}$ is the conjugated complex value of z, and $\varphi_2'(z)$, $\psi_2'(z)$ and $\varphi_2''(z)$ respectively are the first and the second derivatives of the function with respect to z. The functions $\varphi_2(z)$ and $\psi_2(z)$ are analytic complex functions of the complex variable z in the elastic region, i.e. the real and the imaginary part of them satisfy the Cauchy–Riemann conditions (Novozhilov, 1961).

Functions $\varphi_2(z)$ and $\psi_2(z)$ have to be determined by the stress conditions at infinity and by conditions on the elastoplastic interface L (Fig. 1). Let us assume the following boundary conditions:

$$(\sigma_{xx}^\infty+\sigma_{yy}^\infty) = 4a_0+4\mathrm{i}a_1 y \tag{9.1}$$

$$(\sigma_{yy}^\infty-\sigma_{xx}^\infty)+2\mathrm{i}\tau_{xy}^\infty = 2b_0-4\mathrm{i}a_1 y \tag{9.2}$$

$$(\sigma_{xx})^{el} = (\sigma_{xx})^{pl}; \quad (\sigma_{yy})^{el} = (\sigma_{yy})^{pl}; \quad (\tau_{xy})^{el} = (\tau_{xy})^{pl} \text{ on } L. \tag{9.3}$$

The constants a_0 and b_0 represent the biaxial loading and a_1 describes the bending of the plate. Based on the requirement that the stresses and displacements in the elastic area have to be single-valued, Novozhilov (1961) explains that the functions $\varphi_2(z)$ and $\psi_2(z)$ can be written as follows, provided that the point $z = 0$ lies inside the closed loop L:

$$\varphi_2(z) = \varphi_0(z)+a_0 z+a_1\frac{z^2}{2}+(F_x+\mathrm{i}F_y)\lambda_1 \ln z \tag{10.1}$$

$$\psi_2(z) = \psi_0(z)+b_0 z-a_1\frac{z^2}{2}+(F_x+\mathrm{i}F_y)\lambda_2 \ln z. \tag{10.2}$$

Functions $\varphi_0(z)$ and $\psi_0(z)$ are analytic at infinity and can be written as a Laurent series:

$$\varphi_0(z) = \alpha_0 + \alpha_1 \frac{1}{z} + \alpha_2 \frac{1}{z^2} + \alpha_3 \frac{1}{z^3} + \cdots \qquad (11.1)$$

$$\psi_0(z) = \beta_0 + \beta_1 \frac{1}{z} + \beta_2 \frac{1}{z^2} + \beta_3 \frac{1}{z^3} + \cdots \qquad (11.2)$$

where α_0 and β_0 are constants with no influence on the stresses, and λ_1 and λ_2 are constants depending on the material's properties. The coefficients α_j and β_j of the series $\varphi_0(z)$ and $\psi_0(z)$ are calculated from the boundary conditions at the interface L.

The vector $(F_x + iF_y)$ in eqn (10) represents a resulting load applied to the elastoplastic interface. The fact that this resulting load vanishes whenever the stress p acts on the surface of the circular hole is of critical importance. This circumstance was omitted in Galin's solution (1946) for the case where forces are bending the plate and generate an inhomogeneous stress field:

$$\oint_L (f_x + if_y)\, \mathrm{d}s = F_x + iF_y = 0. \qquad (12)$$

2.3. Elastoplastic interface

If the loading path forced a monotonic increase of the yielded zone and all material parameters were continuous, all stresses at the elastoplastic interface L would have to be continuous for reason of equilibrium and continuity of displacements. The validity of such a solution depends on the form of the yielded zone and the distribution of the stress characteristics. The solution is statically determined if any point on the interface L can be connected with the boundary $r = R$ by two stress characteristics lying entirely within the plastic region (Detournay, 1986).

Galin (1946) expresses the stress continuity conditions by means of the biharmonic function U_3:

$$U_3 = U_2 - U_1; \quad \Delta\Delta U_3 = 0 \qquad (13.1)$$

$$\frac{\partial^2 U_3}{\partial x^2} + \frac{\partial^2 U_3}{\partial y^2} = 0; \quad \frac{\partial^2 U_3}{\partial x^2} - \frac{\partial^2 U_3}{\partial y^2} - 2i\frac{\partial^2 U_3}{\partial x\, \partial y} = 0 \quad \text{on } L. \qquad (13.2)$$

2.4. Curvilinear co-ordinates

The mapping function $(x + iy) = \omega(\zeta + i\eta)$ corresponds in its region of regularity to a transformation of the region from Cartesian co-ordinates x, y to curvilinear isometric orthogonal co-ordinates ζ, η (Novozhilov, 1961):

$$z = (x + iy) = \omega(\xi) = \omega(\zeta + i\eta). \qquad (14)$$

By a proper choice of $\omega(\xi)$ the curve L of the elastoplastic interface has the form

$$|\xi| = 1 \quad \text{on } L. \qquad (15)$$

We introduce a mapping function $\omega(\xi)$, which differs from the one in Galin's solution:

$$z = \omega(\xi) = c\xi + c_0 + g\left(\frac{1}{\xi}\right). \qquad (16)$$

It contains an additional term c_0. This additional term allows us to find condition (12), which expresses the integral equilibrium of forces in an explicit way and restricts the functions $\varphi_2(z)$ and $\psi_2(z)$ in eqn (10).

Corresponding to the boundary conditions (9) with the assumed symmetry to the y-axis the real part of c_0 is a floating parameter which can arbitrarily be set to zero, since the

problem is not altered by shifting the hole in the direction of the real axis. This is not the case with the imaginary part of c_0 which is not a floating parameter and has to be determined, since shifting the hole in the direction of the imaginary axis alters the stress field around the hole by the existing stress gradient expressed in eqns (9.1), (9.2).

$g(1/\xi)$ in eqn (16) is an analytic complex function with $\lim_{\xi \to \infty} g(1/\xi) = 0$, where

$$g\left(\frac{1}{\xi}\right) = \sum_{j=1}^{\infty} c_j \frac{1}{\xi^j}. \tag{17}$$

In the curvilinear isometric orthogonal co-ordinate system $\xi = (\zeta + i\eta)$ boundary conditions (9) and continuity conditions (13) can be expressed by the analytic functions $\varphi_3(\xi)$ and $\psi_3(\xi)$ with the following abbreviations:

$$\Phi_k(\xi) = \varphi'_k(\omega(\xi)) = \frac{\varphi'_k(\xi)}{\omega'(\xi)}; \quad \Psi_k(\xi) = \psi'_k(\omega(\xi)) = \frac{\psi'_k(\xi)}{\omega'(\xi)}; \quad (k = 1, 2, 3). \tag{18}$$

Continuity of the mean stress:

$$\mathscr{R}[\Phi_3(\xi)] = \begin{cases} 0 = \mathscr{R}\left[O\left(\frac{1}{\xi^2}\right) + a_0 + a_1 c\xi + a_1 c_0 + a_1 c_1 \frac{1}{\xi}\right] - \frac{k+p}{2} + k \ln R - k \ln |c| \\[2mm] \quad - k \ln |\xi| - k \ln \left|\left(1 + \frac{c_0}{c\xi} + \frac{g(1/\xi)}{c\xi}\right)\right| \quad \text{for } |\xi| = 1 \\[2mm] \mathscr{R}[a_0 + a_1 c\xi + a_1 c_0] - \frac{k+p}{2} + k \ln R - k \ln |c| - k \ln |\xi| \quad \text{for } |\xi| \to \infty. \end{cases}$$

$$\tag{19}$$

Continuity of the deviatoric stress:

$$\overline{\omega(\xi)} \frac{\Phi'_3(\xi)}{\omega'(\xi)} + \Psi_3(\xi) = \begin{cases} 0 & \text{for } |\xi| = 1 \\[2mm] (c\overline{\xi} + \overline{c_0})a_1 + b_0 + b_1(c\xi + c_0) - k\frac{\xi}{\overline{\xi}} & \text{for } |\xi| \to \infty. \end{cases}$$

$$\tag{20}$$

Conditions (19), (20) with $|\xi| = 1$ correspond to the continuity conditions (13) and conditions (19), (20) with $|\xi| \to \infty$ correspond to the boundary conditions at infinity (9.1), (9.2).

In conditions (19) and (20) the stress distribution in the yielded zone is expressed in terms of the functions $\Phi_1(\xi)$ and $\Psi_1(\xi)$ in the curvilinear co-ordinates ξ:

$$\mathscr{R}[\Phi_1(\xi)] = \frac{1}{4}(\sigma_{xx} + \sigma_{yy}) = \frac{k+p}{2} + k \ln \left|\frac{1}{R}\left(c\xi + c_0 + g\left(\frac{1}{\xi}\right)\right)\right| \tag{21}$$

$$\overline{\omega(\xi)} \frac{\Phi'_1(\xi)}{\omega'(\xi)} + \Psi_1(\xi) = \frac{1}{2}(\sigma_{yy} - \sigma_{xx}) + i\tau_{xy} = k \frac{\overline{\omega(\xi)}}{\omega(\xi)}. \tag{22}$$

2.5. Form of the interface

The Laurent series decomposition of $\ln|1 + c_0/c\xi + g(1/\xi)/c\xi|$ with respect to $\xi = 0$ can be calculated along $|\xi| = 1$:

$$\ln\left|1+\frac{c_0}{c\xi}+\frac{g(1/\xi)}{c\xi}\right| = \mathscr{R}\left[\ln\left(1+\frac{c_0}{c\xi}+O\left(\frac{1}{\xi^2}\right)\right)\right] = \mathscr{R}\left[\ln\left(1+\frac{\overline{c_0}}{\bar{c}}\xi+O(\xi^2)\right)\right]$$

$$= \mathscr{R}\left[\ln 1+\frac{\overline{c_0}}{\bar{c}}\xi+O(\xi^2)\right] = \mathscr{R}\left[\frac{c_0}{c}\frac{1}{\xi}+O\left(\frac{1}{\xi^2}\right)\right]. \tag{23}$$

In contrast to Galin's solution, the Laurent series decomposition (23) is taken into account in eqn (19). This is necessary on account of condition (12), which restricts the functions $\varphi_2(z)$ and $\psi_2(z)$ [eqn (10)] by the disappearance of the logarithmic term $\ln z$. The calculation of $\Phi_2(\xi)$ using eqn (18) leads to the expression used in the boundary condition (19) at the elastoplastic interface $|\xi| = 1$. Vanishing of the terms of order $O(1/\xi^2)$ in eqn (19) can be obtained by a proper choice of the free parameters α in eqn (11). This is not the case with the terms of order $O(1/\xi)$, which cannot be adjusted by free parameters without violating the explicit condition for the balance of forces [eqn (12)]. This claim will be explained in more detail in Section 2.6. Expressions containing terms raised to the power of zero or first order must be considered separately; two conditions result from eqn (19) for $|\xi| = 1$:

$$\text{order zero}: \quad \mathscr{R}[a_0+a_1c_0]-\frac{k+p}{2}+k\ln R-k\ln|c| = 0 \tag{24}$$

$$\text{order }\frac{1}{\xi}: \quad \overline{a_1c}+a_1c_1-k\frac{c_0}{c} = 0. \tag{25}$$

Function $\Phi_3(\xi)$ results from eqn (19). Through the proper choice of $\varphi_0(z)$ in eqn (10), all terms raised to the power of second or higher order disappear in eqn (19) for $|\xi| = 1$. Thus for all of eqn (19), $\Phi_3(\xi)$ can be written as:

$$\Phi_3(\xi) = a_1c\xi+a_1c_1\frac{1}{\xi}-k\frac{c_0}{c}\frac{1}{\xi}-k\ln\xi = a_1c\xi-\overline{a_1c}\frac{1}{\xi}-k\ln\xi. \tag{26}$$

Equation (20) for $|\xi| \to \infty$ with expression (26) for the function Φ_3 turns into the following:

$$\bar{z}\Phi_3'(z)+\Psi_3(z) = \overline{\omega(\xi)}\frac{\Phi_3'(\xi)}{\omega'(\xi)}+\Psi_3(\xi) = (\overline{c\xi}+\overline{c_0})\frac{a_1c}{c}-\frac{\bar{c}}{c}k\frac{\xi}{\bar{\xi}}+\Psi_3(\xi)$$

$$= (\overline{c\xi}+\overline{c_0})a_1+b_0+b_1(c\xi+c_0)-k\frac{\xi}{\bar{\xi}} \quad \text{for } |\xi| \to \infty. \tag{27}$$

From this results term $\Psi_3(\xi)$, extended to $|\xi| = 1$,

$$\Psi_3(\xi) = b_0+b_1\left(c\xi+c_0+\frac{c_1}{\xi}\right)+O\left(\frac{1}{\xi^2}\right) \tag{28}$$

and

$$c = \bar{c}. \tag{29}$$

Inserting expressions $\Phi_3(\xi)$ [eqn (26)] and $\Psi_3(\xi)$ [eqn (28)] in eqn (20) for $|\xi| = 1$, we get the equations to determine the mapping function $\omega(\xi)$:

$$\left[c\frac{1}{\xi} + \overline{c_0} + \overline{c_1}\xi + \overline{c_2}\xi^2 + \cdots \right]\left[a_1 c - k\frac{1}{\xi} - \left(a_1 c_1 - k\frac{c_0}{c} \right)\frac{1}{\xi^2} \right]$$

$$= -\left[c - \frac{c_1}{\xi^2} - \frac{2c_2}{\xi^3} - \cdots \right]\left[b_0 + b_1 c_0 + b_1 c\xi + b_1 c_1 \frac{1}{\xi} + O\left(\frac{1}{\xi^2} \right) \right]. \quad (30)$$

If we choose the boundary conditions to be symmetric to the y-axis as was done in eqn (9), then the solution to the problem is also symmetric to the y-axis:

$$a_0 = \overline{a_0}; \quad b_0 = \overline{b_0}; \quad a_1 = -\overline{a_1}. \quad (31)$$

For reasons of symmetry it follows that

$$c_0 = -\overline{c_0}; \quad c_j = \overline{c_j}\,(j = 1, 3, 5, \ldots); \quad c_j = -\overline{c_j}\,(j = 2, 4, 6, \ldots) \quad (32)$$

and from eqn (25) results:

$$a_1 c_1 - k\frac{c_0}{c} = a_1 c. \quad (33)$$

2.6. Violation in Galin's solution

To prove the claim that Galin's solution must be defective in the case where the plate is being bent, we assume that the coefficient $c_0 = 0$ vanishes, as was done by Galin. From eqns (5.2) and (7) and the boundary condition that the mean stress ($\sigma_{xx} + \sigma_{yy}$) is continuous at the interface $|\xi| = 1$ results equation (34):

$$\frac{k+p}{2} + k\ln\left| \frac{\omega(\xi)}{R} \right| = \mathscr{R}[\varphi_2'(z)] = \mathscr{R}\left[\frac{\varphi_2'(\xi)}{\omega'(\xi)} \right]. \quad (34)$$

The Laurent series decomposition of eqn (34) with $|\xi| = 1$ results with respect to eqn (10) in eqn (35):

$$\frac{k+p}{2} - k\ln R + k\ln|c| + k\mathscr{R}\left[O\left(\frac{1}{\xi^2} \right) \right] = \mathscr{R}\left[O\left(\frac{1}{\xi^2} \right) + a_0 + a_1 c\xi + a_1 c_1 \frac{1}{\xi} \right.$$

$$\left. + (F_x + iF_y)\lambda_1 O\left(\frac{1}{\xi} \right) \right] \quad \text{for} \quad |\xi| = 1. \quad (35)$$

Actually, taking into account the equivalence of the coefficients of ξ^{-1} in eqn (35) at the interface $|\xi| = 1$, the result is equation (36):

$$0 = \overline{a_1 c} + a_1 c_1 + (F_x + iF_y)\lambda_1 \cdot \text{const.} \quad (36)$$

From the symmetry conditions (29) and (31) follows:

$$0 = a_1(c_1 - c) + (F_x + iF_y)\lambda_1 \cdot \text{const.} \quad (37)$$

From the boundary condition (9) it follows that in the case of bending the plate the coefficient $a_1 \neq 0$. The vanishing of $(F_x + iF_y)$ can only be achieved if $c = c_1$ as can also be seen in eqn (33) with $c_0 = 0$. Since the equivalence of forces requires the vanishing of $(F_x + iF_y)$ and in Galin's solution $c \neq c_1$, Galin's solution must be defective. If c_0 is set to zero (as in Galin's analysis), there are just not enough degrees of freedom to ensure that an interface can be found such that all the stress components are continuous across it. One aim of the present paper is to look for this solution, in which simultaneous continuity of all stresses in the case of bending is achieved. A numerical example is given in Section 2.8.

2.7. Solution to the equations

In contrast to Galin's solution, we have to bear in mind condition (12) for the integral equilibrium of forces. Therefore we compare the coefficients of all terms raised to the power of zero and positive order in the Laurent's decomposition of eqn (30), as well as the coefficients of the terms raised to the power of first negative order. Together with the conditions (24), (31), (32). (33) and the following system of equations we determine the mapping function $\omega(\xi)$:

$$\xi^{-1}: \quad a_1 cc - k\overline{c_0} - a_1 c_1 \overline{c_1} + kc_0 \frac{\overline{c_1}}{c} = -b_1 c_1 c + c_1 b_1 c \tag{38.-1}$$

$$\xi^0: \quad \overline{c_0} a_1 c - k\overline{c_1} - a_1 c\overline{c_2} = -c(b_0 + b_1 c_0) \tag{38.0}$$

$$\xi^1: \quad a_1 c\overline{c_1} - k\overline{c_2} - a_1 c\overline{c_3} = -b_1 cc \tag{38.1}$$

$$\xi^2: \quad a_1 c\overline{c_2} - k\overline{c_3} - a_1 c\overline{c_4} = 0 \tag{38.2}$$

$$\cdots \qquad \cdots$$

$$\xi^n: \quad a_1 c\overline{c_n} - k\overline{c_{n+1}} - a_1 c\overline{c_{n+2}} = 0. \tag{38.n}$$

Regarding eqns (32), (33), eqn (38.-1) for ξ^{-1} comes about in the same way. Solution (38) for the equations $\xi^2, \xi^3, \ldots$ is already known from Galin's solution (1946) as the solution of a difference system of equations with only one convergent part:

$$\overline{c_n} = Av^n; \quad n \geq 2 \tag{39}$$

with

$$v = -\frac{k}{2a_1 c} - \text{sign}(\mathscr{I}[a_1]) \sqrt{\left(\frac{k}{2a_1 c}\right)^2 + 1}. \tag{40}$$

With the aid of the remaining four equations ξ^0, ξ^1 of (38) and (24), (33), missing parts of the mapping function $\omega(\xi)$ are determined, which differ from that in Galin's solution (1946).

By means of the abbreviations

$$\tilde{\alpha} = a_1 c \tag{41}$$

$$\tilde{\beta} = \frac{(cb_0 - kc)(\tilde{\alpha}v + k)}{(\tilde{\alpha}v + k)(2\tilde{\alpha}^2 + k^2) + \tilde{\alpha}^2 k} \tag{42}$$

we get

$$c = R \exp\left[\frac{1}{k}\left\{\mathscr{R}[a_0 + a_1 c_0] - \frac{k+p}{2}\right\}\right] \tag{43}$$

$$c_1 = c + k\tilde{\beta} \tag{44}$$

$$c_0 = \tilde{\alpha}\tilde{\beta} \tag{45}$$

$$A = \frac{k\tilde{\alpha}\tilde{\beta}}{v^2(\tilde{\alpha}v + k)}. \tag{46}$$

The form of the elastoplastic interface L can be expressed as a function of the boundary conditions at infinity and the stress p that acts upon the wall of the circular hole:

$$L: \quad z = c\xi + c_0 + c_1 \frac{1}{\xi} + \frac{\overline{Av^2}}{\xi(\xi - \bar{v})} \quad \text{with } |\xi| = 1. \tag{47}$$

C(-1)= 2.74573777
C(0)= - .27963607 i
C(1)= - .08320086
C(2)=   .28242299 i
C(3)=   .02819579
C(4)= - .00281506 i
C(5)= - .00028130
C(6)=   .00002811 i
C(7)=   .00000280
C(8)= - .00000027 i
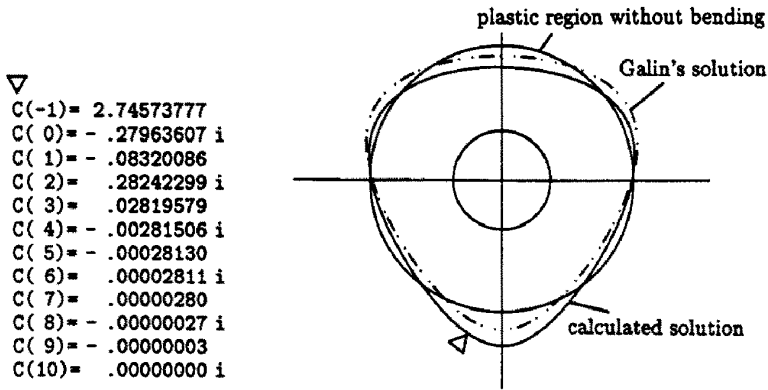C(9)= - .00000003
C(10)=   .00000000 i
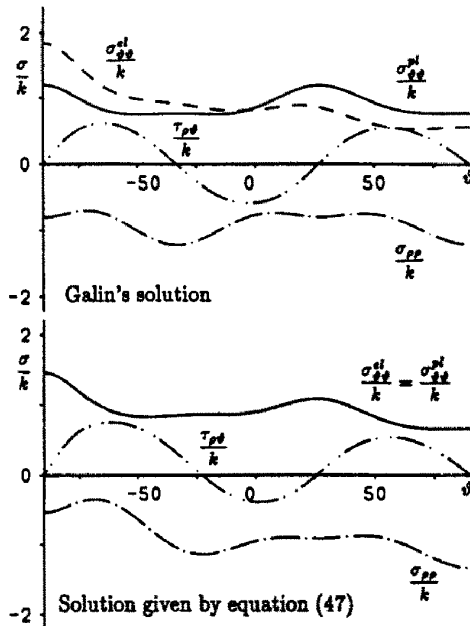
Fig. 2a. Example A of a plate in bending.

Fig. 2b. Stress distribution at the elastoplastic interface in Galin's solution and in the solution given by eqn (47).

2.8. *Example A*

In the first example we compare Galin's solution for the form of the plastic region with the form of the plastic region which is determined by eqn (47). Let us assume the following boundary conditions for Example A:

$$\sigma_{yy}^{\infty} = \tau_{xy}^{\infty} = 0; \quad \sigma_{xx}^{\infty} = -0.1442k\frac{y}{R}; \quad p = -3k; \quad \sigma_{rr}^{\text{plastic}} \leqslant \sigma_{\varphi\varphi}^{\text{plastic}}$$

Starting from this value, we get the form L of the interface:

$$L: \quad \frac{z}{R} = 2.7457\xi - 0.2796i - 0.0832\frac{1}{\xi} + \frac{0.2824i}{\xi(\xi + 0.1000i)}; \quad |\xi| = 1.$$

Galin's solution for the interface L is (Savin, 1956):

$$L: \quad \frac{z}{R} = 2.72\xi - 0i - 0.0266\frac{1}{\xi} + \frac{0.2714i}{\xi(\xi + 0.100i)}; \quad |\xi| = 1.$$

The difference between the solution found and Galin's solution is shown in **Fig. 2a.** The

new calculated interface has been shifted by a certain amount and is slightly deformed in another way.

In Fig. 2b the normal stress $\sigma_{\rho\rho}$, the shear stress $\tau_{\rho\vartheta}$ and the circumferential stress $\sigma_{\vartheta\vartheta}^{el}$ and $\sigma_{\vartheta\vartheta}^{pl}$ at the elastoplastic boundary are plotted with the condition that $F_x + iF_y = 0$. Coordinate transformations of the stresses from polar coordinates (r, φ) to curvilinear coordinates (ρ, ϑ) are performed. It should be demonstrated that actually the circumferential stresses in Galin's solution are discontinuous, if the balance of forces is assumed, in contrast to the correct solution given by eqn (47).

3. EXTENSION TO FRICTIONAL MATERIALS AND DISCONTINUOUS STRESS FIELDS

In Section 3 we will make use of the fundamental equations and methods from Section 2 and explain only modifications of the conditions and the way of solving the problem.

3.1. Plastic region and Coulomb's yield condition

Coulomb's yield condition for a cohesive frictional material is determined by two material parameters: the internal friction angle $\hat{\phi}$ and the cohesion k. Coulomb's condition is an extension of Tresca's condition (2):

$$\sqrt{(\sigma_{xx} - \sigma_{yy})^2 + 4\tau_{xy}^2} = |\sigma_{rr} - \sigma_{\varphi\varphi}| = 2k \cos \hat{\phi} - (\sigma_{rr} + \sigma_{\varphi\varphi}) \sin \hat{\phi}. \tag{48}$$

For the assumed boundary condition (1), the stress distribution in the statically determined plastic zone is axially symmetric. Salençon (1966, 1969) gives the following stress distribution for the two cases $\sigma_{rr} \lessgtr \sigma_{\varphi\varphi}$, if Coulomb's condition (48) is valid:

$$(\sigma_{\varphi\varphi} - \sigma_{rr}) = \frac{\pm 2}{1 \pm \sin \hat{\phi}} \left[k \cos \hat{\phi} - p \sin \hat{\phi} \right] \left(\frac{r}{R} \right)^{\frac{\mp 2 \sin \hat{\phi}}{1 \pm \sin \hat{\phi}}} \tag{49}$$

$$(\sigma_{\varphi\varphi} + \sigma_{rr}) = \frac{2}{1 \pm \sin \hat{\phi}} \frac{-1}{\sin \hat{\phi}} [k \cos \hat{\phi} - p \sin \hat{\phi}] \left(\frac{r}{R} \right)^{\frac{\mp 2 \sin \hat{\phi}}{1 \pm \sin \hat{\phi}}} + 2k \cot \hat{\phi}. \tag{50}$$

3.2. Equilibrium at the elastoplastic interface

Contrary to Galin's (1946) or Detournay's (1986) methods of solving the problem— they both demand continuity of all stresses by assuming eqn (13) (respectively for Detournay in an analog manner)—we examine the equilibrium condition at the elastoplastic interface as a separate important physical condition. Therefore, if we demand that the resultant force $(F_x + iF_y)$ vanishes, we must claim continuity of normal and shear stresses along the elastoplastic interface. This must hold even if the circumferential stress at the elastoplastic interface is not continuous. We calculate the normal and shear stresses along the elastoplastic interface:

$$\sigma_{\rho\rho} + i\tau_{\rho\vartheta} = \tfrac{1}{2}(\sigma_{rr} + \sigma_{\varphi\varphi}) + \tfrac{1}{2}(\sigma_{rr} - \sigma_{\varphi\varphi}) e^{2i(\varphi - \alpha)} \tag{51}$$

where $\sigma_{\rho\rho}$ is the normal stress and $\tau_{\rho\vartheta}$ is the shear stress at the elastoplastic interface in the curvilinear co-ordinates $\xi = \rho e^{i\vartheta}$ defined by eqn (14). The co-ordinates r and φ are the polar co-ordinates introduced and α is the angle between the outward normal of the plastic zone and the x-axis. Angles φ and α can be expressed by the mapping function $\omega(\xi)$ [eqn (14)], with $|\xi| = \rho = 1$:

$$e^{2i\varphi} = \left(\frac{z}{|z|} \right)^2 = \frac{\omega(\xi)}{\overline{\omega(\xi)}} \tag{52}$$

$$e^{-2i\alpha} = \left(\frac{|dz|}{-i \, dz} \right)^2 = \left(\frac{|\omega'(\xi)| \, |d\xi|}{-i\omega'(\xi) \, d\xi} \right)^2 = \left(\frac{|\omega'(\xi)|}{-i\omega'(\xi)i\xi} \right)^2 = \frac{1}{\xi^2} \frac{\overline{\omega'(\xi)}}{\omega'(\xi)}. \tag{53}$$

Continuity of normal and shear stresses towards the elastoplastic interface can be expressed in the curvilinear co-ordinates by the equality of corresponding stress components in the plastic and elastic region. These components are given in the elastic region by the stress functions $\varphi_2(\xi)$, $\psi_2(\xi)$, e.g. Novozhilov (1961). The boundary condition on the elastoplastic interface for surface tractions f_j in the co-ordinate direction j of a Cartesian co-ordinate system can be transformed into the normal and shear stresses $(\sigma_{\rho\rho}+\mathrm{i}\tau_{\rho\vartheta})$ under consideration in the direction of the interface:

$$\varphi_2(\xi)+\frac{\omega(\xi)}{\overline{\omega'(\xi)}}\,\overline{\varphi_2'(\xi)}+\overline{\psi_2(\xi)} = \int_0^s (-f_y+\mathrm{i}f_x)\,\mathrm{d}s = \int_0^s (-f_\vartheta+\mathrm{i}f_\rho)\,\mathrm{e}^{\mathrm{i}\alpha}\,\mathrm{d}s$$

$$= \int_{z_0}^z (-f_\vartheta+\mathrm{i}f_\rho)(-\mathrm{i})\,\mathrm{d}z = \int_1^\xi (\sigma_{\rho\rho}+\mathrm{i}\tau_{\rho\vartheta})\omega'(\xi)\,\mathrm{d}\xi. \quad (54)$$

The integral on the right-hand side of eqn (54) can be solved with the knowledge of the stresses in the plastic region given by eqn (49). We use the following abbreviations and $|\xi| = 1$:

$$K_1 = \frac{\mp 2}{1\pm\sin\hat\phi}(k\cos\hat\phi-p\sin\hat\phi); \quad K_2 = \frac{\mp 1}{\sin\hat\phi}K_1$$

$$K_3 = 2k\cot\hat\phi; \quad \hat\alpha = \frac{\mp 2\sin\hat\phi}{1\pm\sin\hat\phi}; \quad \hat r^2 = \left(\frac{r}{R}\right)^2 = \omega(\xi)\overline{\omega(\xi)} \quad (55)$$

$$\int_1^\xi (\sigma_{\rho\rho}+\mathrm{i}\tau_{\rho\vartheta})\omega'(\xi)\,\mathrm{d}\xi = \int_1^\xi \left[\tfrac{1}{2}(-K_2\hat r^{\hat\alpha}+K_3)+\tfrac{1}{2}(K_1\hat r^{\hat\alpha})\frac{\omega(\xi)}{\omega(\xi)}\frac{1}{\xi^2}\frac{\overline{\omega'(\xi)}}{\omega'(\xi)}\right]\omega'(\xi)\,\mathrm{d}\xi$$

$$= \tfrac{1}{2}K_3\omega(\xi)-\frac{1}{2+\hat\alpha}K_2(\omega(\xi)\overline{\omega(\xi)})^{\hat\alpha/2}\omega(\xi)+\text{const.} \quad (56)$$

3.3. Discontinuous circumferential stresses

Although normal and shear stresses at the elastoplastic interface are continuous on account of the equilibrium conditions, this is not necessarily the case for circumferential stresses. In fact, the circumferential stresses should be continuous if the plastic region expands monotonically and all material parameters are continuous. Discontinuous circumferential stresses (and this is also valid for the circumferential plastic strain not considered here) arise only in the case of unloading, which will not be discussed in this paper, and in the case of discontinuous material parameters. In particular, in Coulomb's yield criterion it is significant to distinguish between the cases concerning the values of the material parameters of cohesion k and internal friction angle $\hat\phi$ in a yielded zone with plastic deformations, and without it. In the same way we distinguish between sticking and slipping friction and between sticking and slipping cohesion. At the elastoplastic interface a small variation Δk of the cohesion k and a small variation $\Delta\hat\phi$ of the internal friction angle $\hat\phi$ may occur, since conditions vary from sticking to slipping by the crossing from the elastic to the plastic region. In the elastic region at the elastoplastic interface the cohesion is $k^{el} = k+\Delta k$, the internal friction angle is $\hat\phi^{el} = \hat\phi+\Delta\hat\phi$ and the circumferential stress is $\sigma_{33}^{el} = \sigma_{33}+\Delta\sigma_{33}$. The values of Δk, $\Delta\hat\phi$ or $\Delta\sigma_{33}$ may be negative. With the aid of Coulomb's yield condition we express the described circumstance in the following manner:

$$k^{el} = k+\Delta k; \quad \hat\phi^{el} = \hat\phi+\Delta\hat\phi; \quad \sigma_{33}^{el} = \sigma_{33}+\Delta\sigma_{33}.$$

Plastic side:

$$F^{\text{slipping}} = \sqrt{(\sigma_{\rho\rho} - \sigma_{3\vartheta})^2 + 4\tau_{\rho3}^2} - 2k \cos \hat{\phi} + (\sigma_{\rho\rho} + \sigma_{3\vartheta}) \sin \hat{\phi} = 0. \qquad (57.1)$$

Elastic side:

$$F^{\text{sticking}} = \sqrt{(\sigma_{\rho\rho} - \sigma_{3\vartheta}^{el})^2 + 4\tau_{\rho3}^2} - 2k^{el} \cos \hat{\phi}^{el} + (\sigma_{\rho\rho} + \sigma_{3\vartheta}^{el}) \sin \hat{\phi}^{el} = 0. \qquad (57.2)$$

The physical consequence of the discontinuous material parameters assumed is far-reaching and interesting. It results in the existence of a finite boundary between the elastic and plastic region, in which energy dissipation takes place. The physics of this boundary can be examined separately, but this is also not a topic of this paper. It is assumed that this finite boundary is thin.

Although the circumferential stress $\sigma_{3\vartheta}$ and the material parameters k, $\hat{\phi}$ may be discontinuous at the elastoplastic interface, the yield function F in eqn (57) is a continuous and smooth function which can be differentiated. Neglecting terms of higher order, since the values Δk, $\Delta \hat{\phi}$ and $\Delta \sigma_{3\vartheta}$ are assumed to be small, the linearized relation between the cohesion, the friction angle and the circumferential stress at the elastoplastic interface is expressed by the tangent to the contour line of the smooth yield function F:

$$\frac{\partial F(k, \hat{\phi}, \sigma_{3\vartheta})}{\partial \sigma_{3\vartheta}} \Delta \sigma_{3\vartheta} + \frac{\partial F(k, \hat{\phi}, \sigma_{3\vartheta})}{\partial k} \Delta k + \frac{\partial F(k, \hat{\phi}, \sigma_{3\vartheta})}{\partial \hat{\phi}} \Delta \hat{\phi} = 0. \qquad (58)$$

If circumferential stresses at the elastoplastic interface are discontinuous, the first invariant of stresses will be discontinuous. To calculate this, we transform the difference between the main stresses:

$$(\sigma_{\rho\rho} - \sigma_{3\vartheta}) = (\sigma_{rr} - \sigma_{\varphi\varphi}) \cos 2 (\alpha - \varphi). \qquad (59)$$

From this we get with the aid of the equations (49), (50), (52), (53), (58) and the continuity of $\sigma_{\rho\rho}$:

$$\Delta(\sigma_{\rho\rho} + \sigma_{3\vartheta}) = \Delta\sigma_{3\vartheta} = \frac{2 \cos \hat{\phi} \, \Delta k - \{2k \sin \hat{\phi} + (-K_2(\omega\bar{\omega})^{\hat{s}/2} + K_3) \cos \hat{\phi}\} \, \Delta\hat{\phi}}{\sin \hat{\phi} - \text{sign}(K_1)\mathscr{R}\left[\dfrac{1}{\xi^2} \dfrac{\omega\bar{\omega}'}{\omega'\bar{\omega}}\right]}$$

$$= D(\omega(\xi)) + \overline{D(\omega(\xi))} = 2\mathscr{R}\left[\sum_{j=0}^{-\infty} D_j(\xi)\xi^j\right] \quad (60)$$

where $D(\omega(\xi))$ is a function of the co-ordinate ξ and describes the discontinuity of the circumferential stress $\Delta\sigma_{3\vartheta}$. The total circumferential stress in the elastic region consists of the part generated by continuous stress fields, and the part generated by discontinuous material parameters, $\sigma_{3\vartheta}^{el} = \sigma_{3\vartheta} + \Delta\sigma_{3\vartheta}$:

$$2\left(\frac{\varphi_2'(\xi)}{\omega'(\xi)} + \frac{\overline{\varphi_2'(\xi)}}{\overline{\omega'(\xi)}}\right) = -K_2(\omega(\xi)\overline{\omega(\xi)})^{\hat{s}/2} + K_3 + D(\xi) + \overline{D(\xi)}. \qquad (61)$$

Detournay (1986) shows that the expression $(\omega\bar{\omega})^{\hat{s}/2}$ can be expanded in a Laurent series using the formula of Miller:

$$(\omega(\xi)\overline{\omega(\xi)})^{\hat{s}/2} = \sum_{j=-\infty}^{\infty} \varepsilon_j \xi^j = (c\bar{c})^{\hat{s}/2} \sum_{j=-\infty}^{\infty} \left\{\sum_{l=0}^{\infty} d_l \overline{d_{l-j}}\right\}\xi^j \qquad (62)$$

where

$$d_0 = 1; \quad d_j = \frac{1}{j} \sum_{l=1}^{\infty} \left[\left(\frac{\hat{\alpha}}{2} + 1 \right) l - j \right] d_{j-l} \frac{c_{l-1}}{c}; \quad j = 1, 2, 3, \ldots, \infty. \tag{63}$$

The stresses on the elastic side of the interface satisfy criterion (57b) with $k + \Delta k$ and $\hat{\phi} + \Delta \hat{\phi}$ as the "sticking" values. This condition, together with the continuity requirement of the stress vector across L and the knowledge of the stress in the plastic zone, constitute a consistent set of conditions to determine L.

To calculate the interface L we need the Laurent series decomposition of eqn (60) which is more complex. To simplify the problem mathematically, we don't claim the variations Δk and $\Delta \hat{\phi}$ to be constant, but rather to be such that $\Delta \sigma_{33} = s = $ const. results [eqn (58)]. Of course Δk and $\Delta \hat{\phi}$ are not uniquely determined by this condition. However, this is not necessary to calculate the stresses in the elastic region, since we know the value of $\Delta \sigma_{33} = s$ a priori. From this results the simple condition:

$$D(\xi) = D_0(\xi) = \tfrac{1}{2}s. \tag{64}$$

It has to be said clearly that the arbitrary assumption (64) is motivated only by mathematical reasons and should be replaced as soon as possible by the physically motivated condition (60). With almost all circular shaped plastic zones, however, the difference between the two conditions is minor.

3.4. Elastoplastic interface

With functions $\varphi_2(\omega(\xi))$ and $\psi_2(\omega(\xi))$ from eqn (10), demanding equivalence of forces [eqn (12)], we get from eqns (54) and (61) the following system of equations. From this system we calculate the form L of the elastoplastic interface:

$$\varphi_0(\xi) + \frac{\omega(\xi)}{\overline{\omega'(\xi)}} \overline{\varphi_0'(\xi)} + \overline{\psi_0(\xi)} = -a_0 \omega(\xi) - \overline{b_0 \omega(\xi)} - \frac{a_1}{2} (\omega(\xi) - \overline{\omega(\xi)})^2$$

$$+ \tfrac{1}{2} K_3 \omega(\xi) - \frac{1}{2 + \hat{\alpha}} K_2 (\omega(\xi) \overline{\omega(\xi)})^{\hat{\alpha}/2} \omega(\xi) + \text{const.} \quad \text{with } |\xi| = 1. \tag{65}$$

$$2 \left(\frac{\varphi_0'(\xi)}{\omega'(\xi)} + \overline{\frac{\varphi_0'(\xi)}{\omega'(\xi)}} \right) = -2a_0 - 2a_1 (\omega(\xi) - \overline{\omega(\xi)})$$

$$- K_2 (\omega(\xi) \overline{\omega(\xi)})^{\hat{\alpha}/2} + K_3 + D(\xi) + \overline{D(\xi)} \quad \text{with } |\xi| = 1. \tag{66}$$

The imaginary part of eqn (66) vanishes. Additionally, the left-hand side of eqn (66) doesn't have terms raised to the power of zero and first order as a consequence of eqn (10) and the mapping function $\omega(\xi)$ [eqn (16)]. From the last we get two conditions, that the terms raised to the power of zero and first order on the right-hand side of eqn (66) must balance [(70.1) and (70.2)]. To briefly express this we use the following abbreviation for the decomposition of functions:

$$F_n \left[\sum_{j=-\infty}^{\infty} a_j \xi^j \right] = a_n. \tag{67}$$

Since all terms raised to the power of zero or positive order in eqn (65) can be balanced by expression $\overline{\psi_0(\xi)}$, we get a system of equations for the coefficients of all terms with negative exponents. To determine the mapping function $\omega(\xi)$ we express the function $\varphi_0(\xi)$ by $\omega(\xi)$ from eqn (66) with abbreviation (67):

$$\varphi_0'(\xi) = -a_1 \sum_{j=-2}^{-\infty} F_j[\omega(\xi) - \overline{\omega(\xi)}]\xi^j \omega'(\xi) - \tfrac{1}{2}K_2\omega'(\xi) \sum_{j=-2}^{-\infty} \varepsilon_j\xi^j + \tfrac{1}{2}\omega'(\xi) \sum_{j=-2}^{-\infty} D_j(\xi)\xi^j.$$

$$(68)$$

With the two equations from (66) for the terms raised to the power of zero and first order and the equations for the coefficients of the terms with negative exponents in eqn (65), we get in respect of expression (68) a nonlinear system of equations for the coefficients of the mapping function $\omega(\xi)$ [eqn (16)]. With the assumption that $\omega(\xi)$ can be expressed by a finite Laurent series,

$$\omega(\xi) = \sum_{j=-1}^{n} c_j \frac{1}{\xi^j}; \quad c_{-1} = c$$

$$(69)$$

we get the following system:

$$-2a_0 - 2a_1(c_0 - \overline{c_0}) - K_2\varepsilon_0 + K_3 + D_0 = 0$$

$$(70.1)$$

$$-2a_1(c_1 - \overline{c_{-1}}) - K_2\varepsilon_1 + D_1 = 0$$

$$(70.2)$$

for $m = 1, 2, 3, \ldots, \infty$

$$\sum_{j=-1}^{n} \left\{ -a_1 \frac{1}{m}[c_j j c_{m-j}]_{\{2 \leqslant m-j \leqslant n\}} - \tfrac{1}{2}K_2 \frac{1}{m}[c_j j \varepsilon_{m-j}]_{\{2 \leqslant m-j\}} + \frac{1}{2} \frac{1}{m}[c_j j D_{m-j}]_{\{2 \leqslant m-j\}} \right.$$

$$-\overline{a_1}[c_j \overline{c_{j-m}}]_{\{2 \leqslant j-m \leqslant n\}} - \tfrac{1}{2}K_2[c_j \overline{\varepsilon_{j-m}}]_{\{2 \leqslant j-m\}} + \tfrac{1}{2}[c_j \overline{D_{j-m}}]_{\{2 \leqslant j-m\}}$$

$$+ \tfrac{1}{2}a_1[c_j c_{m-j}]_{\{-1 \leqslant m-j \leqslant n\}} - a_1[c_j \overline{c_{j-m}}]_{\{-1 \leqslant j-m \leqslant n\}} + \tfrac{1}{2}a_1[\overline{c_j c_{-j-m}}]_{\{-1 \leqslant -j-m \leqslant n\}}$$

$$\left. + \frac{1}{2+\hat{\alpha}}K_2[c_j \overline{\varepsilon_{m-j}}] \right\} + 2a_0 c_m + \overline{b_0 c_{-m}} - \tfrac{1}{2}K_3 c_m = 0.$$

$$(70.3)$$

The solution of the infinite nonlinear system of equations (70) gives the coefficients of the mapping function $\omega(\xi)$ [eqn (16)]. An approximate solution can be found by expressing the mapping function in a finite series [eqn (69)], and to minimize the error in a finite number of equations of the system [eqn (70)]. A mathematical proof that this approximation method converges towards the sought solution is not known to the author. But we can assume that the fewer the terms by which the series of the mapping function $\omega(\xi)$ is truncated, the fewer constraints are acting on the form of the calculated elastoplastic interface. Therefore the approximated solution should converge to the sought one. These assumptions are confirmed by numerical results in such a way that the coefficients of the truncated series $\omega(\xi)$ converge numerically to the analytical solution (47) for Tresca's yield criterion. An analytical solution of the problem for a material with infinite internal friction ($\hat{\phi} \to \pi/2$) is given by Cherepanov (1963, 1964), and can also be confirmed numerically as a limit case of the equations presented in this paper.

4. EXAMPLES

The following examples present only a limited number of various parameter combinations. The examples are by no means complete and will only give an idea of the different forms of plastic regions. The examples presented are valid up to limited deviations of the plastic zones from a circular shape. The limits are discussed by Detournay (1986). Also, a minor discontinuity of the circumferential stress is assumed. To demonstrate the influence of the parameters on the form of the plastic zones clearly, an exaggerated variation of the parameters without regarding the limits is performed.
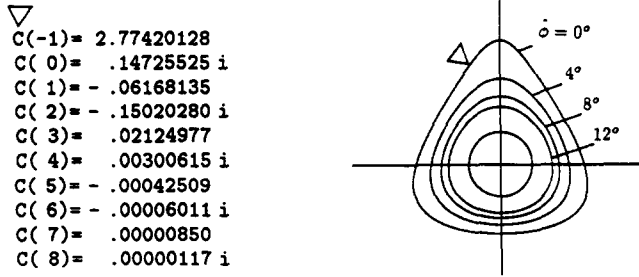
$\triangledown$
C(-1)= 2.77420128
C(0)=   .14725525 i
C(1)= - .06168135
C(2)= - .15020280 i
C(3)=   .02124977
C(4)=   .00300615 i
C(5)= - .00042509
C(6)= - .00006011 i
C(7)=   .00000850
C(8)=   .00000117 i

Fig. 3. Example B: compression and bending of a plate, with various internal friction angles $\hat{\phi}$.

4.1. *Example B*

Let us assume the following boundary conditions for Example B and examine the geometry of the elastoplastic interface for different internal friction angles $\hat{\phi}$:

$$\tau_{xy}^{\infty} = 0; \quad \sigma_{yy}^{\infty} = -3k; \quad \sigma_{xx}^{\infty} = -3k - 0.2k\frac{y}{R}, \quad p = 0; \quad \sigma_{rr}^{\text{plastic}} \geqslant \sigma_{\varphi\varphi}^{\text{plastic}}.$$

In this example the plastic region around the circular hole is a result of a pressure $(-3k)$ at infinity and a bending of the plate $(\partial\sigma_{xx}^{\infty}/\partial y = -0.2k/R)$. How significant the influence of the internal friction angle $\hat{\phi}$ on the shape and magnitude of the elastoplastic interface is can be seen in Fig. 3. By reducing the internal friction angle $\hat{\phi}$, the magnitude of the plastic zone increases and its shape varies from a circular-like to a triangular-like form.

4.2. *Example C*

In Example C the internal friction angle is $\hat{\phi} = 30°$. The following boundary conditions are considered:

$$\tau_{xy}^{\infty} = 0; \quad \sigma_{xx}^{\infty} = \sigma_{yy}^{\infty} = -10k, \, p = 0; \quad \sigma_{rr}^{\text{plastic}} \geqslant \sigma_{\varphi\varphi}^{\text{plastic}}.$$

An overlayed bending of the plate causes a variation of the shape of the plastic zone. Since unloading of the plastic region is not admissible, the trajectories shown are not achievable by increasing the bending. They can, however, be reached on separate loading paths (compare Detournay, 1986), Fig. 4.

4.3. *Example D*

In Example D similar conditions to Example C are observed, except that stresses at infinity are different in the x- and y-directions. The shape of the plastic zone reminds one of a deformed ellipse:
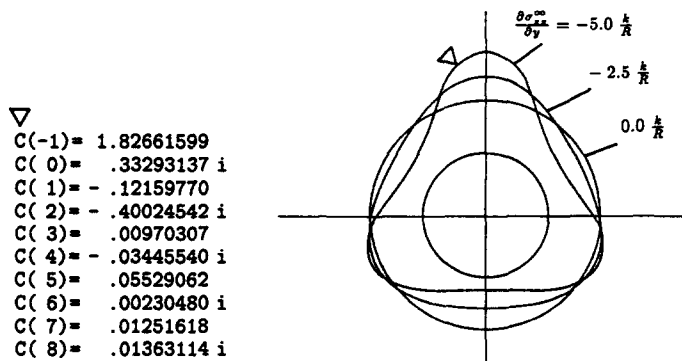
$\triangledown$
C(-1)= 1.82661599
C(0)=   .33293137 i
C(1)= - .12159770
C(2)= - .40024542 i
C(3)=   .00970307
C(4)= - .03445540 i
C(5)=   .05529062
C(6)=   .00230480 i
C(7)=   .01251618
C(8)=   .01363114 i

Fig. 4. Example C: compression and bending of a plate, internal friction angle $\hat{\phi} = 30°$.

$$\frac{\partial \sigma_{xx}^{\infty}}{\partial y} = -2.5\,\frac{k}{R}$$
$$-1.25\,\frac{k}{R}$$
$$0.0\,\frac{k}{R}$$

C(-1)= 1.96734593
C(0)= .21978299 i
C(1)= .28493818
C(2)= - .20148083 i
C(3)= - .03317069
C(4)= .03504852 i
C(5)= .01919035
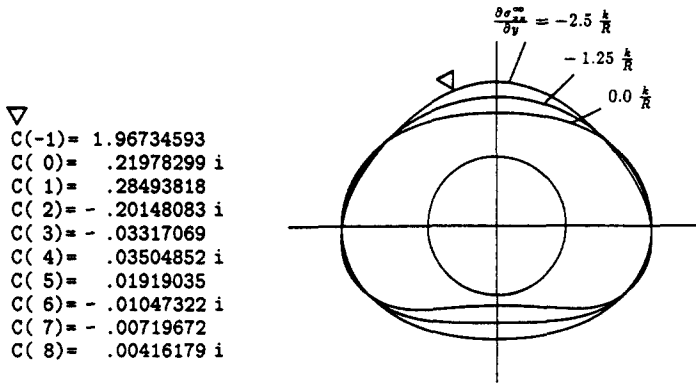C(6)= - .01047322 i
C(7)= - .00719672
C(8)= .00416179 i

Fig. 5. Example D: biaxial compression of a plate and overlayed bending, internal friction angle $\phi = 30°$.

$$\tau_{xy}^{\infty} = 0; \quad \sigma_{xx}^{\infty} = -10k; \quad \sigma_{yy}^{\infty} = -14k, \, p = 0; \quad \sigma_{rr}^{plastic} \geqslant \sigma_{\varphi\varphi}^{plastic}.$$

An overlayed bending of the plate causes the variation of the shape of the plastic zone shown. In comparison to Example C, trajectories are not reachable on only one loading path, since unloading is not admissible; Fig. 5.

4.4. *Example E*

Example E shows a biaxially pressed plate. The internal friction angle is $\hat{\phi} = 30°$. Boundary conditions are:

$$\tau_{xy}^{\infty} = 0; \quad \sigma_{xx}^{\infty} = -10k; \quad \sigma_{yy}^{\infty} = -14k, \, p = 0; \quad \sigma_{rr}^{plastic} \geqslant \sigma_{\varphi\varphi}^{plastic}.$$

Discontinuous circumferential stresses at the elastoplastic interface decisively influence the shape of the plastic region. The circumferential stress in the elastic region at the elastoplastic interface is higher than the circumferential stress in the plastic region at the interface. The difference is a constant amount. By increasing this stress difference the plastic region shrinks (this is not to be understood as an unloading path) and changes its form as shown in Fig. 6.

In the course of examination of the elastoplastic interface around a circular hole, experiments with rectangular specimens made of cohesive frictional material were carried out (Tokar, 1987). The material used was Araldidt B, an epoxi resin of the Ciba Company, Germany, with a special amount (4% by weight) of the hardener PSA (Burgert, 1981; Kuch, 1988). This material is transparent in the elastic regime. Yielding of the materials in triaxial tests was examined by Kuch (1988) using cylindrical specimens, showing yielding according to Coulomb's condition. In the experiments performed (Tokar, 1987), a triaxial loading was applied to rectangular specimens containing a circular hole. Around the circular hole a plastic region developed. In Fig. 7 one specimen is shown, containing a plastified region with a geometrical form similar to the calculated one shown in Fig. 6. Whether
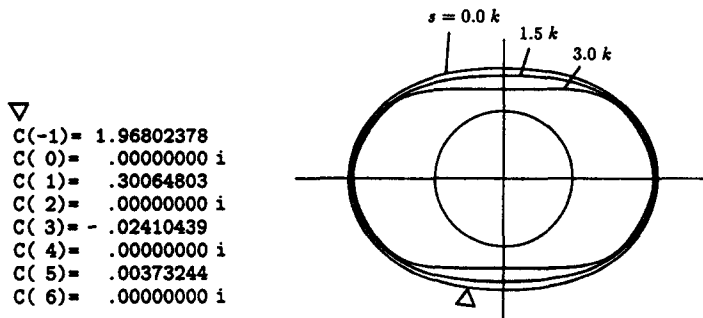
$$s = 0.0\,k$$
$$1.5\,k$$
$$3.0\,k$$

C(-1)= 1.96802378
C(0)= .00000000 i
C(1)= .30064803
C(2)= .00000000 i
C(3)= - .02410439
C(4)= .00000000 i
C(5)= .00373244
C(6)= .00000000 i

Fig. 6. Example E: biaxial compression of a plate, internal friction angle $\hat{\phi} = 30°$; discontinuous circumferential stress at the elastoplastic interface.
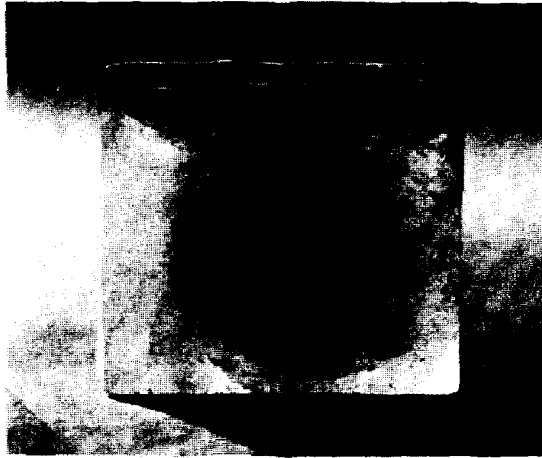
Fig. 7. Biaxial compression of a specimen made of cohesive frictional material indicates a plastic zone enclosing the circular hole.

the stresses at the elastoplastic interface in the experiment were in fact continuous or discontinuous (as assumed in the calculation) cannot be determined from the tests performed.

5. CONCLUDING REMARKS

The form of the plastic zone around a circular hole in an inhomogeneous stress field has been discussed. Plane strain conditions and a plastic zone enclosing the hole were assumed. The inhomogeneous stress field represents a biaxial loading and an overlayed bending, as illustrated in Fig. 1. Yielding of the material is described by Tresca's and Coulomb's yield condition. Both continuity and discontinuity of the circumferential stresses at the elastoplastic boundary were considered.

The main achievements of the paper are the corrected analytical solution of the form of the plastic zone around a circular hole in an inhomogeneous stress field for Tresca's material and a quasi-analytic solution for the case with Coulomb's material. The quasi-analytical solution in the second case is an analytic description of the problem in a nonlinear infinite system of equations and an approximative numerical solution to the equations.

The significance of the results are the calculation of the failed zone around a long cylindrical underground cavity in an inhomogeneous stress field, as well as with the possibility of either continuous or discontinuous stresses at the elastoplastic interface. The forms of the calculated plastic zones possess complex shapes, and are illustratd in the examples. The results are limited to a certain amount of deviation of the geometrical form of the elastoplastic interface from the circular shape. The limits are discussed by Detournay (1986) and Detournay and Fairhurst (1987).

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